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# AGM-Style Contraction in Dung-Logics for Argumentation Frameworks

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# Contents

**Abstract**

This thesis investigates the possibility of implementing an AGM-style contraction operator for argumentation frameworks. We enhance the works of Baumann and Brewka [1] who defined a family of monotonic Dung-logics on argumentation frameworks for which they implemented both AGM-style expansion and revision operators. The ordinary equivalence of these logics coincides with strong equivalence in the ordinary sense. The AGM-Theory of belief revision defined contraction operators axiomatically via postulates. We rephrase these postulates to make them applicable to Dung-logics and argumentation frameworks. We then can show that there is no contraction operator on these monotonic Dung-logics. However, we reflect on two concepts for contraction operators and show why they are not satisfying all contraction postulates. After having completed these considerations, we investigate how the $k$-revision operator given by Baumann and Brewka [1] relates to one of our *contraction* operators. In the conclusion we consider further possibilities of achieving contraction, i. e. changing the logic or changing the postulates.

# 1 Introduction

When dealing with the theory of *intelligent agents* sooner or later one will come upon the problem of *belief representation*. An agent as an abstract entity, dedicated to reaching certain goals and equipped with certain sensors, needs *some* way to represent knowledge about its environment in order to meaningfully decide what to do next, respectively to act.

In the subject of belief representation we will consider two concepts each of which touches a different aspect of it, namely *abstract argumentation* and the *AGM-Theory of Belief Revision* (named by its founders Alchourrón, Gärdernfors and Makinson [2]). Whilst the former concept is about actual representation of (possibly) conflicting beliefs, the latter handles the question of what to do when beliefs are subject of change.

This thesis aims at enhancing on the works of Baumann and Brewka in [1]. In their research paper they combined aforementioned concepts of abstract argumentation and AGM-theory of belief revision. They presented two operators that implement AGM-style expansion and revision on a a family of logics of argumentation frameworks, the so-called *Dung-Logics*. We will expand their results by investigating the possibilities and limitations of an operator that implements AGM-style contraction on Dung-logics.

Before approaching this problem we will present the background of this thesis followed by an overview of Baumann's and Brewka's groundwork.

# 2 Background

## 2.1 Abstract Argumentation

At its heart, the theory of abstract argumentation - invented by P. H. Dung in 1995 [3] - is about representing conflicts between pieces of knowledge. The core concept of abstract argumentation is the *argumentation framework (AF)* which can be understood as a directed graph. An AF consists of two types of elements: arguments (nodes) and attacks (edges). An argument represents a piece of knowledge whereas an attack represents a conflict between two arguments. AFs do not hold information about the essence of arguments or reasons for why certain arguments conflict with each other. But the theory of abstract argumentation does supply ways of reasoning with AFs. Using a concept called *semantics*, one can decide which selection of arguments is consistent in regards to this semantics.

In what follows, we will present the core concepts of abstract argumentation accompanied by examples. We will conclude this section by giving an overview on how the concepts we present are of use in the example of knowledge representation for intelligent agents as outlined in section 1.

### 2.1.1 Argumentation Frameworks

**Definition 1.** An AF is a tuple $F = (A, R)$ where $A$ is the set of arguments, a finite subset of a fixed infinite background set $\mathcal{U}$ and $R$ is the set of *attacks*, a subset of $A \times A$. By $\mathscr{A}$ we denote the set of all AFs. We write $A(F)$ for $A$ and $R(F)$ for $R$.

**Definition 2.** Given two AFs $F, G \in \mathscr{A}$ we define each usual set-theoretical operator $\circ \in \{\subseteq, \subset, =\}$ and $\cup$ component-wise:

$$F \circ G \Leftrightarrow A(F) \circ A(G) \wedge R(F) \circ R(G) \tag{1}$$

$$F \cup G = \big(A(F) \cup A(G), R(F) \cup R(G)\big) \tag{2}$$

**Definition 3.** Given an AF $F \in \mathscr{A}$, $a, b \in A(F)$ we say:

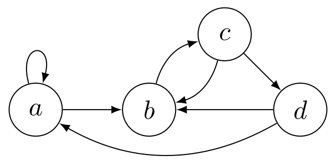
(1) $a$ *attacks* $b$ iff $(a, b) \in R(F)$.

(2) $a$ is *defended* by $A' \subseteq A(F)$ iff $\forall b \in A(F) : (b, a) \in R(F) \Rightarrow \exists c \in A' : (c, b) \in R(F)$.

**Definition 4.** Given an AF $F$, for some $A' \subseteq A(F)$, $A'^{+}$ denotes $A' \cup \{b \mid (a, b) \in R(F) \wedge a \in A'\}$. We say $A'$ *covers* $a \in A$ iff $a \in A'^{+}$.

**Definition 5.** A set $A' \subseteq A(F)$ is *conflict-free* iff $R(F) \cap (A' \times A') = \emptyset$. The set of conflict free sets of $F$ is denoted by $cf(F)$.

**Example.** Let us inspect $F_{1.1}$ in figure 1 in detail. There, $a$ is attacked by itself and $d$ whereas $b$ is attacked by $a$, $c$ and $d$. Thus $c$ is defended by any set in $2^{\{a,c,d\}}$ that is not empty. All conflict-free sets of arguments are given by: $\emptyset$, $\{b\}$, $\{c\}$ and $\{d\}$.



$F_{1.1}$

Figure 1: AF Example

### 2.1.2 Semantics and Extensions

The notion of *consistent sets of arguments* as mentioned in the introduction of this chapter is expressed by various types of so called extensions. An *extension* is a subset of arguments of an AF that is consistent in regards to a certain semantics. A *semantics* is a function which assigns to any AF $F \in \mathscr{A}$ a set of extensions: $\sigma : \mathscr{A} \to 2^{2^{\mathcal{U}}}$, $\sigma(F) \subseteq 2^{A(F)}$. We consider nine semantics namely admissible ($ad$), complete ($co$), preferred ($pr$), semi-stable ($ss$), stable ($stb$), stage ($stg$), grounded ($gr$), ideal ($id$) and eager ($eg$).

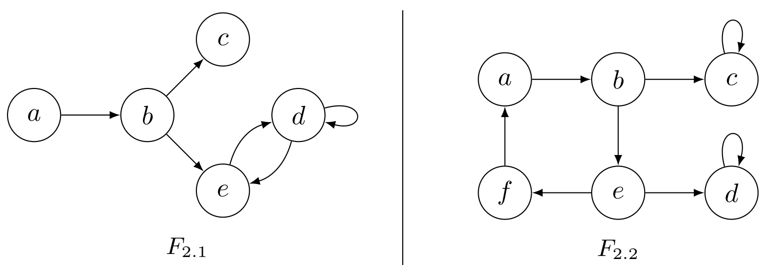



Figure 2: AF Examples for Semantics

In what follows, we will define the so-called $\sigma$-extensions $E \in \sigma(F)$ of $F = (A, R) \in \mathscr{A}$ with $\sigma \in \{ad, co, pr, ss, stb, stg, gr, id, eg\}$ and $E \subseteq A$. For each definition, we will give an example by using the AFs displayed in figure 2.

**Definition 6.** An *admissible*-extension is conflict-free and defends all of its elements:

$$E \in ad(F) \Leftrightarrow E \in cf(F) \land \forall a \in E : E \text{ defends } a \tag{3}$$

**Example.** When looking at $F_{2.1}$ one can observe that $d$ will not be part of any admissible-extension. In this example, all admissible-extensions are given by: $\emptyset$, $\{a\}$, $\{a, e\}$, $\{a, c\}$, $\{a, c, e\}$. For $F_{2.2}$, neither $c$ nor $d$ can be part of any admissible extension. In this example, all admissible-extensions are given by: $\emptyset$, $\{a, e\}$, $\{b, f\}$.

**Definition 7.** A *complete*-extension is an admissible-extension and includes all arguments which are defended by the extension:

$$E \in co(F) \Leftrightarrow E \in ad(F) \land \forall a \in A : E \text{ defends } a \Rightarrow a \in E \tag{4}$$

**Example.** For $F_{2.1}$, all complete-extensions are given by: $\{a, c\}$, $\{a, c, e\}$. For $F_{2.2}$, all complete-extensions are given by: $\emptyset$, $\{a, e\}, \{b, f\}$. In this case, the empty set is a complete-extension because there is no argument not attacked by another. By these examples you can see that demanding of an extension to include all defended arguments does not simply lead to a selection of $\subseteq$-maximal admissible-extensions.

**Definition 8.** A *preferred*-extension is an admissible-extension that is $\subseteq$-maximal:

$$E \in pr(F) \Leftrightarrow E \in ad(F) \wedge \forall E' \in ad(F) : E \not\subset E' \tag{5}$$

**Example.** Preferred-extensions can easily be read off by inspecting the admissible-extensions. Looking at the admissible-extensions of $F_{2.1}$, we see, there is only one preferred-extension: $\{a, c, e\}$. For $F_{2.2}$, all preferred-semantics are given by: $\{a, e\}$, $\{b, f\}$.

**Definition 9.** A *semi-stable*-extension is an admissible-extension that is $\subseteq$-maximal in terms of its covered arguments:

$$E \in ss(F) \Leftrightarrow E \in ad(F) \wedge \forall E' \in ad(F) : E^+ \not\subset E'^+ \tag{6}$$

**Example.** Again, looking at all admissible-extensions of $F_{2.1}$, we can figure out the only semi-stable-extension $\{a, c, e\}$, with $\{a, c, e\}^+ = \{a, b, c, d, e\}$. For $F_{2.2}$, all semi-stable-extensions are given by: $\{a, e\}$, $\{b, f\}$, each of which covers all arguments but $c$ or $d$ respectively.

**Definition 10.** A *stable*-extension is an admissible-extension that covers all arguments, i. e. is the $\subseteq$-greatest admissible-extension in terms of its covered arguments if existent:

$$E \in stb(F) \Leftrightarrow E \in cf(F) \wedge E^+ = A \tag{7}$$

**Example.** Since we already noticed that $\{a, c, e\}$ covers all arguments it is also the only stable-extension of $F_{2.1}$. For $F_{2.2}$, there is no stable-semantics. We already mentioned that each semi-stable-extension of $F_{2.2}$ misses out on covering one argument. Therefore neither of them is supremal in this regards.

**Definition 11.** A *stage*-extension is conflict-free and $\subseteq$-maximal in terms of its covered arguments, similar to preferred-extensions:

$$E \in stg(F) \Leftrightarrow E \in cf(F) \wedge \forall E' \in cf(F) : E^+ \not\subset E'^+ \tag{8}$$

**Example.** For $F_{2.1}$, there is only one stage-extension: $\{a, c, e\}$. Also for $F_{2.2}$, the stage-extensions are the same as the preferred-extensions: $\{a, e\}$, $\{b, f\}$. But note: This is not necessarily the case as it is not required of a stage-extension to defend its elements.

**Definition 12.** A *grounded*-extension is a complete-extension which is $\subseteq$-minimal:

$$E \in gr(F) \Leftrightarrow E \in co(F) \wedge \forall E' \in co(F) : E' \not\subset E \tag{9}$$

**Example.** The grounded-extension is a counterpart to preferred-extensions. A grounded-extension is required to be a $\subseteq$-minimal complete-extension and is always uniquely defined. Therefore we can find the grounded-extensions by simply looking at the complete-extensions. For $F_{2.1}$, the grounded-extension is given by: $\{a, c\}$. For $F_{2.2}$, the grounded extension is given by: $\emptyset$.

**Definition 13.** An *ideal*-extension is an admissible-extension which is bounded by the intersection of all preferred-extensions and is $\subseteq$-maximal:

$$\begin{aligned} E \in id(F) \Leftrightarrow\ & E \in ad(F) \\ & \wedge E \subseteq \bigcap pr(F) \\ & \wedge \forall E' \in ad(F) : E' \subseteq \bigcap pr(F) \Rightarrow E \not\subset E' \end{aligned} \tag{10}$$

**Example.** Just like grounded-extensions, an ideal-extension is always uniquely defined. Since there is only one preferred-extension which happens to be an admissible-extension as well, the ideal-extension of $F_{2.1}$ is: $\{a, c, e\}$. For $F_{2.2}$, the intersection of all preferred-extensions is empty. Therefore we end up with the ideal-extension being: $\emptyset$.

**Definition 14.** An *eager*-extension is an admissible-extension which is bounded by the intersection of all semi-stable-semantics and is $\subseteq$-maximal:

$$\begin{aligned} E \in eg(F) \Leftrightarrow\ & E \in ad(F) \\ & \wedge E \subseteq \bigcap ss(F) \\ & \wedge \forall E' \in ad(F) : E' \subseteq \bigcap ss(F) \Rightarrow E \not\subset E' \end{aligned} \tag{11}$$

**Example.** The eager-semantics can be viewed as a variation of the ideal-semantics and again is uniquely defined. This means there is also only one eager-extension for $F_{2.1}$: $\{a, c, e\}$. For $F_{2.2}$ the eager-extension is given by: $\emptyset$.

### 2.1.3 An Argumentation Framework Use Case

Given a semantics $\sigma$ and an AF $F$ that represents knowledge about an environment, an agent can select *consistent* (in regards to $\sigma$) extensions of $F$ to restrict the space of what is possible about the world to assume. This is useful if the agent has to deal with many and conflicting pieces of knowledge. Argumentation frameworks do not allow for advanced reasoning techniques beyond that, i. e. they can not reason about what *is* true. The theory of abstract argumentation also does not imply a way to gauge the quality of an extension or the quality of a semantics. This is subject of further theories and research or subject of choice.

## 2.2 Monotonic Dung-Logics

The concept of semantics as just presented induces a logic that is implicit to the theory of abstract argumentation.

**Definition 15.** Given a semantics $\sigma$ and an AF $F$ we define the models of $F$ as:

$$Mod^{\sigma}(F) = \sigma(F) \tag{12}$$

An argumentation framework $F \in \mathscr{A}$ then implies another argumentation framework $G \in \mathscr{A}$ iff $Mod^{\sigma}(F) \subseteq Mod^{\sigma}(G)$.

**Definition 16.** We say $F$ and $G$ are ordinary equivalent, i. e. $F \equiv^{\sigma} G$, iff:

$$Mod^{\sigma}(F) = Mod^{\sigma}(G) \tag{13}$$

**Definition 17.** We say $F$ and $G$ are strongly equivalent, i. e. $F \equiv^{\sigma}_{E} G$, iff:

$$\forall H \in \mathscr{A} : Mod^{\sigma}(F \cup H) = Mod^{\sigma}(G \cup H) \tag{14}$$

### 2.2.1 Kernels

Taking a naive approach, the problem of strong equivalence, i. e. to decide whether two AFs are strongly equivalent, is a co-semidecidable problem. One would have to iterate over all AFs and test whether strong equivalence holds for each AF iterated to check whether two AFs are not strongly equivalent. However, this inefficiency can be addressed by the concept of kernels.

**Definition 18.** Given an AF $F \in \mathscr{A}$, a *kernel* is a function $k : \mathscr{A} \rightarrow \mathscr{A}$ where each $k(F) = F^k = \left(A, R^k\right)$ is obtained from $F$ by deleting certain redundant attacks. We call $F$ *k-r-free* iff $F = F^k$.

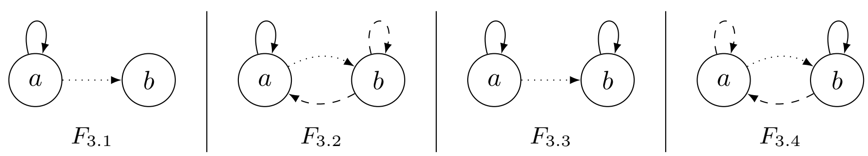



Figure 3: Kernel examples

Given a semantics $\sigma \in \{stb, ad, gr, co\}$ we define $\sigma$-kernels $k(\sigma)(F) = F^{k(\sigma)} = \left(A, R^{k(\sigma)}\right)$:

**Definition 19.** The *stable-kernel* has all attacks removed that are outgoing from an argument with a self-loop:

$$R^{k(stb)} = R \setminus \{(a,b) \mid a \neq b \wedge (a,a) \in R\} \tag{15}$$

**Example.** In figure 3 you can see four examples each of which will be used to illustrate one of the kernel definitions. The dotted arrows indicate kernel-redundant attacks to be deleted whereas the dashed arrows represent optional attacks of which at least one must be present. $F_{3.1}$ illustrates the base case to apply a stable-kernelization to. Since a stable-extension must be conflict free, $a$ can not be part of any complete-extension. Furthermore it is required of a stable-extension to cover all arguments. Therefore any attack not outgoing from an argument, that is part of a conflict-free set, does not contribute to any stable-extension. This is why those attacks might be deleted safely in regards to stable-extensions.

**Definition 20.** The *admissible-kernel* has all attacks removed that are outgoing from an argument with a self-loop but only if the other argument involved has a self-loop as well or there is a respective reverse attack:

$$R^{k(ad)} = R \setminus \{(a,b) \mid a \neq b \wedge (a,a) \in R \wedge \{(b,a),(b,b)\} \cap R \neq \emptyset\} \tag{16}$$

**Example.** By $F_{3.2}$ you can see an illustration of admissible-kernalization. Again, an admissible-extension must be conflict-free therefore $a$ can not be part of any admissible-extension. The attack $(a,b)$ might be safely deleted because either $\{b\}$ is not conflict-free thus can not be part of an admissible-extension or $b$ defends itself against $a$.

**Definition 21.** The *complete-kernel* has all attacks removed that connect two arguments with a self-loop:

$$R^{k(co)} = R \setminus \{(a,b) \mid a \neq b \wedge (a,a),(b,b) \in R\} \tag{17}$$

**Example.** $F_{3.3}$ is an example for complete-kernelization which is straightforward. A complete-extension is an admissible-extension that includes all elements which are defended by itself. Neither $a$ nor $b$ can be part of any admissible- or complete-extension since both have a self-loop. Therefore we do not need to care about any attacks which take place among them.

**Definition 22.** The *grounded-kernel* has all attacks removed that are incoming to an argument with a self-loop but only when the other argument involved as a self-loop as well or there is a respective reverse attack:

$$R^{k(gr)} = R \setminus \{(a,b) \mid a \neq b \wedge (b,b) \in R \wedge \{(a,a),(b,a)\} \cap R \neq \emptyset\} \tag{18}$$

**Example.** $F_{3.4}$ illustrates a grounded-kernelization which is somehow inverse to the admissible-kernelization. Remember, a grounded-extension is a $\subseteq$-minimal complete-extension. If the self-loop $(a,a)$ is part of the AF, we end up with the same situation as described in the example for definition 21. If $(b,a)$ is part of the AF and $(a,a)$ is not, removing $(a,b)$ leads to a change of the complete-extensions. In the original AF $\{a\}$ is a complete-extension, in the modified AF this is not the case anymore. But in both cases $\emptyset$ remains to be a complete-extension and is the $\subseteq$-minimal complete extension as well. This could be extended to larger AFs because removing $(a,b)$ from the AF would not touch the minimal complete-extension.

**Theorem 1** ([4], [5])**.** *For a semantics $\sigma$ and two AFs $F$ and $G$, strong equivalence coincides with kernel identity:*

*(1)* $F \equiv^{\sigma}_{E} \Leftrightarrow F^{k(\sigma)} = G^{k(\sigma)}$ *if* $\sigma \in \{stb, ad, co, gr\}$

*(2)* $F \equiv_E^\sigma \Leftrightarrow F^{k(ad)} = G^{k(ad)}$ *if* $\sigma \in \{pr, id, ss, eg\}$

*(3)* $F \equiv_E^\sigma \Leftrightarrow F^{k(stb)} = G^{k(stb)}$ *if* $\sigma = stg$

This insight allows to efficiently decide whether two AFs are strongly equivalent. Now one must just iterate over all attacks for each AF to compute the respective kernels after which a simple comparison for equality decides over strong equivalence.

### 2.2.2 A Family of Dung-Logics

Equipped with this new tool to decide on strong equivalence we define a logic whose ordinary equivalence coincides with strong equivalence as introduced in definition 17. We define a family of monotonic logics, so-called Dung-logics, by introducing the notion of a *k-model* which in turn determines an abstract consequence relation $\models^k$ constituting $\mathcal{L}^k_{Dung} = (\mathscr{A}, \models^k)$.

Intuitively, a $k$-model of an AF $F \in \mathscr{A}$ is any AF which satisfies at least the information of $F$ not including redundancy, but may have more information than encoded by $F$.

**Definition 23.** Given a kernel $k$, an AF $F \in \mathscr{A}$ and a set of AFs $\mathcal{F} \subseteq \mathscr{A}$, the *set of k-models* is defined as:

$$Mod^k(F) = \left\{G \in \mathscr{A} \mid F^k \subseteq G^k\right\} \tag{19}$$

$$Mod^k(\mathcal{F}) = \bigcap_{F \in \mathcal{F}} Mod^k(F) \tag{20}$$

**Definition 24.** Given a set of AFs $\mathcal{F} \subseteq \mathscr{A}$:

(1) $\mathcal{F}$ is *k-satisfiable* iff $Mod^k(\mathcal{F}) \neq \emptyset$.

(2) $\mathcal{F}$ is *k-tautological* iff $Mod^k(\mathcal{F}) = \mathscr{A}$.

**Definition 25.** The *k-consequence relation* $\models^k \subseteq 2^{\mathscr{A}} \times \mathscr{A}$ is defined as follows:

$$\mathcal{F} \models^k G \Leftrightarrow Mod^k(\mathcal{F}) \subseteq Mod^k(G) \tag{21}$$

**Definition 26.** The *k-consequence operation* $Cn^k : 2^{\mathscr{A}} \to 2^{\mathscr{A}}$ is given by:

$$\mathcal{F} \mapsto Cn^k(\mathcal{F}) = \{G \in \mathscr{A} \mid \mathcal{F} \models^k G\} \tag{22}$$

**Definition 27.** We say two AFs $F, G \in \mathscr{A}$ are *k-equivalent* iff they have the same models:

$$F \equiv^k G \Leftrightarrow Mod^k(F) = Mod^k(G) \tag{23}$$

**Theorem 2** ([1])**.** *Given two AFs F and G k-equivalence coincides with strong equivalence:*

*(1)* $F \equiv^{k(\sigma)} G \Leftrightarrow F \equiv_E^\sigma G$ *if* $\sigma \in \{stb, ad, co, gr\}$

*(2)* $F \equiv^{k(ad)} G \Leftrightarrow F \equiv^{\sigma}_{E} G$ *if* $\sigma \in \{pr, id, ss, eg\}$

*(3)* $F \equiv^{k(stb)} G \Leftrightarrow F \equiv^{\sigma}_{E} G$ *if* $\sigma = stg$

**Definition 28.** A *Dung-logic* $\mathcal{L}^k_{Dung}$ is given by $(\mathscr{A}, \models^k)$ or $(\mathscr{A}, Cn^k)$, respectively.

Although we just defined the family of Dung-Logics on *sets* of AFs, in the following sections we will implement AGM-style operators on this logic only for single AFs - not for sets of AFs. Therefore, in what follows we will drop braces for any $F \in \mathscr{A}$ whenever we write $\{F\} \models^k G$, $G \in \mathscr{A}$ or $Cn^k(\{F\})$. This also leads to two insights which will be of use in later passages:

**Lemma 1.** *Given two AFs* $F, G \in \mathscr{A}$*:*

$$F \models^k G \Leftrightarrow G^k \subseteq F^k \tag{24}$$

*Proof.*

$$\begin{aligned}
F \models^k G &\Leftrightarrow Mod^k(F) \subseteq Mod^k(G) && \text{by definition 25} && (25)\\
&\Leftrightarrow \forall H \in Mod^k(F) : H \in Mod^k(G) && && (26)\\
&\Rightarrow F^k \in Mod^k(G) && \text{since } F^k \in Mod^k(F) && (27)\\
&\Rightarrow G^k \subseteq F^k && \text{by definition 23} && (28)
\end{aligned}$$

$$\begin{aligned}
G^k \subseteq F^k &\Rightarrow \forall F' \in \mathscr{A} : F^k \subseteq F'^k \Rightarrow G^k \subseteq F'^k && && (29)\\
&\Rightarrow \forall F'^k \in Mod^k(F) : F'^k \in Mod^k(G) && \text{by definition 23} && (30)\\
&\Rightarrow Mod^k(F) \subseteq Mod^k(G) && && (31)\\
&\Leftrightarrow F \models^k G && \text{by definition 25} && (32)
\end{aligned}$$

□

**Lemma 2.** *Given an AF* $F \in \mathscr{A}$*, then* $\{F\}$ *is k-satisfiable.*

*Proof.*

$$\begin{aligned}
Mod^k(\{F\}) &= \bigcap_{F' \in \{F\}} Mod^k(F') && \text{by definition 23} && (33)\\
&= Mod^k(F) && && (34)\\
&= \{G \in \mathscr{A} \mid F^k \subseteq G^k\} && \text{by definition 23} && (35)\\
&= \{F^k\} \cup \{G \in \mathscr{A} \mid F^k \subset G^k\} && && (36)\\
&\neq \emptyset && && (37)
\end{aligned}$$

□

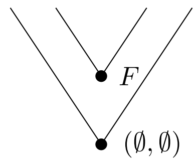



Figure 4: Structure of monotonic Dung-Logics

All Dung-logics are structurally similar. They include the same syntactical and semantical elements which are both AFs. And they all induce the same structure of their elements as illustrated in figure 4. There you can see the $k$-tautological hence empty AF $(\emptyset, \emptyset)$ spanning the space of all $k$-models with an arbitrary AF $F \in \mathscr{A}$ spanning its subspace of $k$-models. Any AF $F' \in \mathscr{A}$ covering the cone of $F$ would be a consequence of $F$, e. g. in particular $(\emptyset, \emptyset)$ is a consequence of $F$: $F \models^k (\emptyset, \emptyset)$. Although this figure successfully illustrates the $\subseteq$-relation of AFs and their models, it still is just an illustration and therefore inappropriate in a way. The figure conveys the impression that there are two dimensions by which the space of models is spanned. This is not the case. One could argue for *arguments* being the one and *attacks* being the other dimension but first, the attacks are dependent on the arguments and second, there is no clear order on those "dimensions". Also, this illustration suggests that for any AFs $F, G \in \mathscr{A}$ there is a non-empty intersection of their models, i. e. $\{F, G\}$ is $k$-satisfiable, which is not the case. There are indeed $k$-unsatisfiable sets of AFs - even with two elements. But therefore it is important to note that although two AF's cones might intersect in an illustration, this intersection can actually be empty. We will still make use of this kind of illustrations since they are suitable to convey key concepts of this thesis but we wanted to also make sure the boundaries of these figures are known.

**Example.** In figure 5, there are two AFs given for which the set $\mathcal{F}$ containing both of them, is $k$-unsatisfiable. This holds for every kernel presented in this thesis but we will present this example for the complete-kernel $k(co)$.

First, note that both $F_{5.1}$ and $F_{5.2}$ are $k(co)$-r-free. Any model $F \in Mod^{k(co)}(\mathcal{F})$ must possess all elements of $F_{5.1}$ and $F_{5.2}$. But no $k(co)$-$r$-free AF can possess the attack $(a, a)$, $(b, b)$ and $(b, a)$ at the same time since kernelization would remove $(b, a)$ from the AF. Therefore $\mathcal{F}$ is $k$-unsatisfiable.

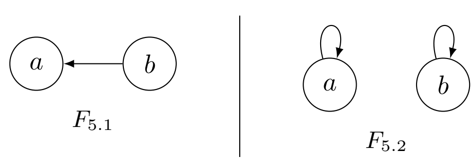



Figure 5: $k$-unsatisfiable AFs [1]

## 2.3 AGM-Theory of Belief Revision

Whereas the theory of abstract argumentation was about *representing* conflicts between pieces of knowledge, the *AGM-Theory of Belief Revision* [2] (in short AGM-Theory) is about *resolving* conflicts between pieces of knowledge which is represented by so called *belief sets*. A belief set $K$ is a set of propositional formulas that is closed under deduction, i. e. $Cn(K) = \{\phi \mid K \models \phi\} = K$. The AGM-Theory was contrived to solve the problem of updating existent beliefs which arose because propositional logic is monotonic.

**Definition 29.** Given a logic $\mathcal{L} = (\mathscr{U}, Cn)$ with a set of syntactical elements $\mathscr{U}$ and a consequence operator $Cn$, $\mathcal{L}$ is monotonic iff for any $\Gamma, \Delta \subseteq \mathscr{U}$ one has:

$$\Gamma \subseteq \Delta \Rightarrow Cn(\Gamma) \subseteq Cn(\Delta) \tag{38}$$

In short, monotony means that no matter what we expand our set of formulas with, we will always be able to deduce those formulas which have already been deducible before expanding the aforementioned set of formulas. When updating beliefs, it would therefore not be suitable to simply expand our belief set with new beliefs as they could never remove information from the belief set.

Let us consider an example belief set $K = Cn(\{\phi, \phi \to \psi, \psi\})$ where $\phi$ means *It rained recently* and $\psi$ means *The street is wet*. What should we do to our belief set if we walked out only to find out the street *is not* wet? We could simply *expand* our belief set by adding $\neg\psi$: $K = Cn(\{\phi, \phi \to \psi, \psi, \neg\psi\})$, but this would lead to an inconsistency in our belief set since both $K \models \psi$ and $K \models \neg\psi$. And no matter what we expanded our belief set with, if at some point we would have $\neg\psi$ in it, this inconsistency would appear. The AGM-Theory tackled this problem by introducing two new operations on belief sets: *revision* and *contraction*. These two operations were introduced via postulates that define what must hold for a resulting belief set. This means, there is no *unique right* way to revise or contract a belief set and thus, there have been many different implementations of these operators for propositional logic as well as for other logics.

In this section, we will introduce the contraction and revision operators as developed by Brewka and Baumann for Dung-logics. In a later section we will approach belief set contraction for Dung-Logics, the core problem of this thesis. Before we come to that, we have to redefine the concept of belief sets to match Dung-logics.

**Definition 30.** A $k$-belief set is an AF $F \in \mathscr{A}$.

We apply the AGM-Theory to Dung logics because the problem of updating belief sets is applicable for Dung-logics as they are monotonic as well.

**Theorem 3.** *Any Dung-logic $L^k_{Dung} = (\mathscr{A}, Cn^k)$ is monotonic, i. e. for any $\mathcal{F}, \mathcal{G} \subseteq \mathscr{A}$ one has*

$$\mathcal{F} \subseteq \mathcal{G} \Rightarrow Cn^k(\mathcal{F}) \subseteq Cn^k(\mathcal{G}) \tag{39}$$

*Proof.*

$$
\begin{aligned}
\mathcal{F} \subseteq \mathcal{G} \Rightarrow & \forall F \in \mathcal{F} : F \in \mathcal{G} && (40)\\
& \Rightarrow Mod^k(\mathcal{G}) = \bigcap_{F \in \mathcal{G}} Mod^k(F) \\
& \qquad\qquad = \bigcap_{F \in \mathcal{F}} Mod^k(F) \cap \bigcap_{F \in \mathcal{G} \setminus \mathcal{F}} Mod^k(F) \\
& \qquad\qquad = Mod^k(\mathcal{F}) \cap Mod^k(\mathcal{G} \setminus \mathcal{F}) && \text{by definition 23} \quad (41)\\
& \Rightarrow Mod^k(\mathcal{G}) \subseteq \Big( \bigcap_{F \in \mathcal{F}} Mod^k(F) \Big) = Mod^k(\mathcal{F}) && (42)\\
& \Rightarrow \forall H \in \mathscr{A} : Mod^k(\mathcal{F}) \subseteq Mod^k(H) \\
& \qquad\qquad \Rightarrow Mod^k(\mathcal{G}) \subseteq Mod^k(H) && (43)\\
& \Rightarrow \forall H \in \mathscr{A} : \mathcal{F} \models^k H \Rightarrow \mathcal{G} \models^k H && \text{by definition 25} \quad (44)\\
& \Rightarrow \forall H \in \mathscr{A} : H \in Cn^k(\mathcal{F}) \Rightarrow H \in Cn^k(\mathcal{G}) && \text{by definition 26} \quad (45)\\
& \Rightarrow Cn^k(\mathcal{F}) \subseteq Cn^k(\mathcal{G}) && (46)
\end{aligned}
$$

□

Let us inspect this result in more detail as there is some tension between definition 30 and theorem 3. Whereas monotony for propositional logic lead to the problem of what to do when updating beliefs as the belief sets themselves are monotonic in regards to the consequence operator $Cn$, this can not simply be transferred to $k$-belief sets because not them but sets of $k$-belief sets are monotonic in regards to the consequence operator $Cn^k$. We can however transfer monotony under application of $Cn^k$ to $k$-belief sets partially:

**Proposition 1.** *For any AFs $F, G \in \mathscr{A}$ one has*

$$F^k \subseteq G^k \Rightarrow Cn^k(F) \subseteq Cn^k(G) \qquad (47)$$

*Proof.*

$$
\begin{aligned}
F^k \subseteq G^k \Rightarrow & G \models^k F && \text{by lemma 1} \quad (48)\\
& \Rightarrow Mod^k(G) \subseteq Mod^k(F) && \text{by definition 25}\\
& \Rightarrow \forall H \in \mathscr{A} : Mod^k(F) \subseteq Mod^k(H) \\
& \qquad\qquad \Rightarrow Mod^k(G) \subseteq Mod^k(H) && (49)\\
& \Rightarrow \forall H \in \mathscr{A} : F \models^k H \Rightarrow G \models^k H && \text{by definition 25} \quad (50)\\
& \Rightarrow \forall H \in \mathscr{A} : H \in Cn^k(F) \Rightarrow H \in Cn^k(G) && \text{by definition 26} \quad (51)\\
& \Rightarrow Cn^k(F) \subseteq Cn^k(G) && (52)
\end{aligned}
$$

□

Generally one does not have $F \subseteq G \Rightarrow Cn^k(F) \subseteq Cn^k(G)$ which can be shown by a simple counter example. This example can be derived directly from one of the illustrations for kernelization in figure 3. Consider the AFs $F_{6.1}$ and $F_{6.2}$ as illustrated in figure 6. Note that $F_{6.1}$ is *k(ad)-r*-free but $F_{6.2}$ is not as $(a, b)$ would be removed during kernelization. Obviously it applies that $F_{6.1}$ is a sub-AF of $F_{6.2}$ but this does not hold for their respective kernels. Therefore we have $F_{6.2} \notin Cn^{k(ad)}(F_{6.1})$ but $F_{6.2} \in Cn^{k(ad)}(F_{6.2})$ which leads to $Cn^{k(ad)}(F_{6.1}) \nsubseteq Cn^{k(ad)}(F_{6.2})$.

This counter example however is applicable only to attacks that play a role in kernelization as any other attack is covered by proposition 1. This means that one can not generally *remove* information from an AF by adding new elements to it which is the key "problem" of monotony causing the necessity of the operators introduced by the AGM-Theory.

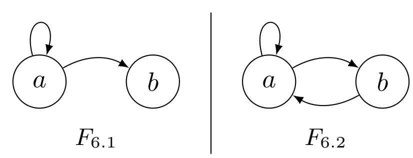



Figure 6: Counter example illustration

#### 2.3.1 Expansion

In the AGM-Theory, the expansion operator is essential in the way that both revision and contraction refer to it. The expansion operator initially was defined semantically. It was required of the expansion result to possess the intersection of the two belief sets $K_1$ and $K_2$ as its own models. This was reflected by the union of $K_1$ and $K_2$. For Dung-logics one then has:

**Definition 31.** A function $\circ^k : \mathscr{A} \times \mathscr{A} \to \mathscr{A} \cup \{\perp\}$ with $(F, G) \mapsto F \circ^k G$ is a *k-expansion* iff $Mod^k(F \circ^k G) = Mod^k(F) \cap Mod^k(G)$.

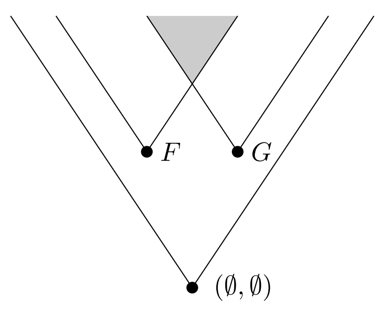



Figure 7: Dung-logic expansion

The set of models sought when expanding $F \in \mathscr{A}$ with $G \in \mathscr{A}$ is illustrated in figure 7 as grey-shaded area. The expansion operator should result in the AF indicated by the intersection point of the cones of $F$ and $G$. In most cases, $k$-expansion is straight-forward and can be given by the union of $F$ and $G$ but in some cases, there is no AF that the $k$-expansion can result in. This is the case whenever $\{F, G\}$ is $k$-unsatisfiable, i. e. when $Mod^k(F) \cap Mod^k(G) = \emptyset$, as an $k$-unsatisfiable AF does by lemma 2 not exist. We map to a new symbol $\perp$ that denotes the $k$-unsatisfiability of the expansion result in this case.

**Definition 32.** The function $+^k : \mathscr{A} \times \mathscr{A} \to \mathscr{A} \cup \{\perp\}$ is a $k$-expansion operator and defined as:

$$(F, G) \mapsto \begin{cases} \perp & \text{if } Mod^k(F) \cap Mod^k(G) = \emptyset \\ F^k \cup G^k & \text{else} \end{cases} \tag{53}$$

Let us reflect on why this method works for any AFs $F, G \in \mathscr{A}$ given $Mod^k(F) \cap Mod^k(G) \neq \emptyset$. Baumann and Brewka [1] were able to show that $Mod^k(F) \cap Mod^k(G) \neq \emptyset \Leftrightarrow F^k \cup G^k$ is $k$-$r$-free. This means that all elements of $F^k$ and $G^k$ are preserved when they are unified into one AF. This also means that whenever we have an AF $H \in Mod(F +^k G)$ we know that $H^k \subseteq F^k \cup G^k$ as we also have $H \subseteq F^k$ and $H \subseteq G^k$. This is why we can map $F +^k G$ to $F^k \cup G^k$.

### 2.3.2 Revision

The intuition behind revising two belief sets can not be conveyed as simply as the intuition behind the expansion operator. What comes closest to the idea of revision is the notion of *updating* a belief set with a new belief without leading to inconsistencies. Given a propositional belief set $K$ and a new belief $\phi$ we want to update our belief set with, we ask from the result of a revision operator $*$ to deduce $\phi$ and to *not* deduce $\neg\phi$: $K * \phi \not\models \phi \wedge K * \phi \models \phi$.

As already mentioned in the introduction, the revision and contraction operators were not defined invariably but outlined by postulates. We will not give a complete list of all postulates established for the revision operator but skip directly to the implementation of a revision operator $*^{k(stb)}$ and the ideas that establish it. For an overview over the postulates of belief revision we refer the reader to the works of Baumann and Brewka [1].

When revising $F \in \mathscr{A}$ with $G \in \mathscr{A}$, the concept that underlies this operator is to find a $\subseteq$-maximal AF "between" $G^k$ and $G^k \cup F^k$. One could also say revising $F$ with $G$ means to expand $G$ with the $\subseteq$-maximal AF $H$ in $F$ that leads to a consistent AF. This allows for preserving as much information as possible from $F$ whilst including $G$ in total. Such an $\subseteq$-maximal AF $H$ is given by the concept of maximal $k$-$r$-free sets.

**Definition 33.** Given two AFs $F, G \in \mathscr{A}$ we define the set of *maximal k-r-free sets* w.r.t. $F$ and $G$ as follows:

$$\mathcal{M}^k_{FG} = \max_{\subseteq}\{G^k \cup H \mid H \subseteq F^k \wedge G^k \cup H \text{ is } k\text{-r-free}\} \tag{54}$$

It has already been shown by Baumann and Brewka [1] that for any AFs $F, G \in \mathscr{A}$, $|\mathcal{M}_{FG}^{k(stb)}| = 1$.

**Definition 34.** The function $*^{k(stb)} : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is a *k(stb)*-revision operator and defined as:

$$(F, G) \mapsto I,\ \mathcal{M}_{FG}^{k(stb)} = \{I\} \tag{55}$$

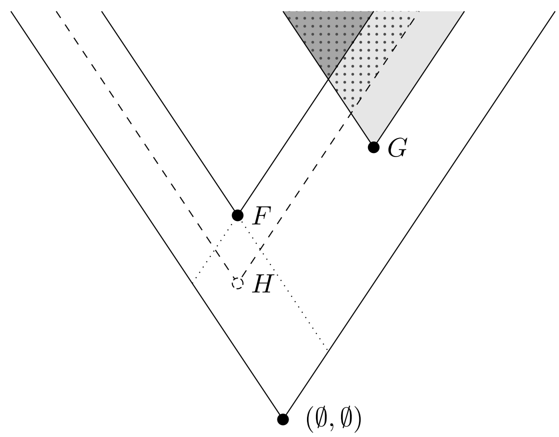



Figure 8: Dung-logic revision

Figure 8 illustrates the revision of $F$ with $G$. There you can see the space of possible sets leading to a $\subseteq$-maximal extension of $G$ marked by dotted lines below $F$. An example for $H$ is given by dashed lines. In this example, the space of models resulting from $G^{k(stb)} \cup H$ or $F *^{k(stb)} G$ is marked by the dotted area. The models of $F *^{k(stb)} G$ cover at least the area colored in dark-gray and at maximum the area colored in light-gray.

### 2.3.3 Towards Expansion and Revision Operators for Belief *Sets*

The reader might have stumbled over the fact that our definition of belief sets involves AFs only. This might seem counterintuitive because our belief sets do not include syntactical elements but *are* syntactical elements of the respective logic. This however is not as counterintuitive as it might seem. The idea behind belief sets is to match *theories*. It can be argued about what exactly a *theory* is in regards to abstract argumentation. But it is not unreasonable to take an AF as theory as we have shown in section 2.1.3 where we presented an application of AFs. AFs encompass the conflicts between various pieces of knowledge and therefore might be taken as a *theory* themselves. This notion is also part of the formalization of AFs where the arguments form a *set*.

But it is possible - to some extent - to widen the definition of belief sets such that it matches sets of AFs. Before we come to that, we will present a key insight in the structure of Dung-logis.

**Proposition 2** ([1]). *Any infinite set of AFs $\mathcal{F} \subseteq \mathscr{A}$ is k-unsatisfiable.*

*Proof.* Let us assume there is a $k$-satisfiable, infinite set of AFs $\mathcal{F} \subseteq \mathscr{A}$. Since $\mathcal{F}$ is $k$-satisfiable we have $Mod^k(\mathcal{F}) \neq \emptyset$.

Given an arbitrary $G \in Mod(\mathcal{F})$ we have:

$$
\begin{aligned}
& G \in Mod(\mathcal{F}) && && (56)\\
\Leftrightarrow\ & G \in \bigcap_{F \in \mathcal{F}} Mod(F) && \text{by definition 23} && (57)\\
\Leftrightarrow\ & \forall F \in \mathcal{F} : G \in Mod(F) && && (58)\\
\Leftrightarrow\ & \forall F \in \mathcal{F} : F^k \subseteq G^k && \text{by definition 23} && (59)
\end{aligned}
$$

By definition 1 we know that AFs are finite. This means that for any AF $F \in \mathscr{A}$ the set $F_{sub} = \{G \in \mathscr{A} \mid G^k \subseteq F^k\}$ is finite as it is bounded by the cartesian product of $2^{A(F)}$ and $2^{R(F)}$, both of which are finite as well. Looking at (59) we therefore can now that $\mathcal{F}$ is finite which contradicts our assumption. □

This insight allows us to phrase a new expansion operator on sets of AFs $\tilde{+}^k : 2^{\mathscr{A}} \times \mathscr{A} \to 2^{\mathscr{A}}$:

$$
(\{F_1, ..., F_n\}, G) \mapsto \begin{cases} F_1 +^k ... +^k F_n +^k G & \text{if } Mod(\{F_1, ..., F_n, G\}) \neq \emptyset \\ \mathcal{F} \cup \{G\} & \text{else} \end{cases} \quad (60)
$$

This is possible only because we know that $\mathcal{F}$ is finite when $Mod(\mathcal{F}) \neq \emptyset$ which allows us to gradually expand its elements by one another. If $\mathcal{F}$ was infinite thus unsatisfiable it would not be possible to "sum" the elements of $\mathcal{F}$ "up". The result of the expansion would need to possess the same set of models as $\mathcal{F}$ which is impossible as by lemma 2, a single AF is always satisfiable.

The idea of this new expansion operator can now be used to define a *partial k-revision* operator $\tilde{*}^k : 2^{\mathscr{A}} \times \mathscr{A} \to 2^{\mathscr{A}}$:

$$
(\mathcal{F}, G) \mapsto \begin{cases} (\mathcal{F}_1 +^k ... +^k \mathcal{F}_n) *^k G & \text{if } Mod(\mathcal{F}) \neq \emptyset \\ \text{undefined} & \text{else} \end{cases} \quad (61)
$$

Further elaboration must be spent on what to do when revising $k$-unsatisfiable sets since they can not be mapped to single AFs which are always $k$-satisfiable. This section however showed that the $k$-expansion and $k$-revision operators of Baumann and Brewka are expressive already.

# 3 Contraction Postulates

The contraction operator implements the idea of *rejecting* something what was formerly accepted as true. In this section we will list the six postulates for propositional logic belief sets which have been phrased in the AGM-Theory to define contraction operators and which aimed at modeling the *real world* procedure of rejecting a belief. Then we will rephrase each one of them to make them applicable to Dung-logics. Before doing this, we will reflect on a characteristic of sets of propositional formulas closed under $Cn$ which will allow us to adequately rephrase those postulates.

**Lemma 3.** *Given two sets of propositional formulas* $\Gamma, \Delta$ *with* $\Gamma = Cn(\Gamma)$ *and* $\Delta = Cn(\Delta)$ *one has*

$$\Gamma \subseteq \Delta \Leftrightarrow \Delta \models \Gamma \tag{62}$$

*Proof.*

$$\begin{aligned} \Gamma \subseteq \Delta &\Rightarrow \forall \phi \in \Gamma : \phi \in \Delta && (63)\\ &\Rightarrow \forall \phi \in \Gamma : \Delta \models \phi && (64)\\ &\Rightarrow \Delta \models \Gamma && (65) \end{aligned}$$

$$\begin{aligned} \Delta \models \Gamma &\Rightarrow \forall \phi \in \Gamma : \Delta \models \phi && && (66)\\ &\Rightarrow \forall \phi \in \Gamma : \phi \in \Delta && \text{by assumption } \Delta = Cn(\Delta) && (67)\\ &\Rightarrow \Gamma \subseteq \Delta && && (68) \end{aligned}$$

□

Given a belief set $K$, formulas $\phi$ and $\psi$ and an expansion operator $+$, there are six postulates:

**K1** $K \div \phi$ is a belief set *(closure)*

**K2** $K \div \phi \subseteq K$ *(inclusion)*

**K3** $\phi \notin K \Rightarrow K \div \phi = K$ *(vacuity)*

**K4** $\not\models \phi \Rightarrow \phi \notin K \div \phi$ *(success)*

**K5** $K \subseteq (K \div \phi) + \phi$ *(recovery)*

**K6** $\models (\phi \leftrightarrow \psi) \Rightarrow K \div \phi = K \div \psi$ *(equivalence)*

Given the AFs $F, G, H \in \mathscr{A}$ we can rephrase the AGM contraction postulates for a kernel $k$.

Lemma 1 shows us that in general for $k$-belief sets one does not have $F \subseteq K \Leftrightarrow K \models^k F$ or $F = G \Leftrightarrow F \equiv^k G$ both of which hold for belief sets which you can see by lemma 3. This means we need to translate each postulate involving some relation $\sim \in \{\in, \subseteq, =\}$ to their equivalent logical relation in $\{\models, \equiv\}$.

The *closure* postulate demands nothing more than that the result is again a belief set. Hence we can rephrase it to:

**K1** $F \div^k G$ is an AF

The *inclusion* postulate demands of a contraction to never add information. Lemma 3 allows us to rephrase this postulate to use the $\models$ relation $K \div \phi \subseteq K \Leftrightarrow K \models K \div \phi$ which in turn allows us to apply this postulate to $k$-belief sets:

**K2** $F \models^k F \div^k G$ which by lemma 1 equals $(F \div^k G)^k \subseteq F^k$

The *vacuity* postulate demands of a contraction to not change the belief set if the belief set to be contracted is not entailed by the original belief set. $\phi \notin K \Rightarrow K \div \phi = K$ equals $K \not\models \phi \Rightarrow K \div \phi \equiv K$ which can directly be adopted for $k$-belief sets:

**K3** $F \not\models^k G \Rightarrow F \div^k G \equiv^k F$

The *success* postulate demands of a contraction to actually remove the contracted element's information from the belief set, meaning that it should not be deducible from the resulting belief set. This is only restricted to be not applicable to the removal of a tautology since a tautology is always deducible. The equivalent in terms of logical relations to $\not\models \phi \Rightarrow \phi \notin K \div \phi$ is $\not\models \phi \Rightarrow K \div \phi \not\models \phi$. Therefore the translation of this postulate to match $k$-belief sets is:

**K4** $\not\models^k G \Rightarrow F \div^k G \not\models^k G$ which equals $\not\models^k G \Rightarrow G^k \not\subseteq (F \div^k G)^k$

The *recovery* postulate demands that the belief set resulting from an expansion with a formerly contracted element must possess at least the information that was present before given element was contracted from the belief set. The intuition behind this postulate was to ensure minimal contraction. However, there were many people who pointed out this postulate was not ideal in this regard and furthermore was problematic in itself. In section 4 you will see why this postulate is problematic for Dung-logics and $k$-belief sets and in section 7 we will discuss this postulate and its issues more in detail.

For $K \subseteq (K \div \phi) + \phi$ lemma 3 directly applies, therefore we end up with $(K \div \phi) + \phi \models K$ which translates to:

**K5** $(F \div^k G) +^k G \models^k F$ which equals $F^k \subseteq \big((F \div^k G) +^k G\big)^k$

The *equivalence* postulate demands of a contraction to result in the same belief set if performed with equivalent elements. Under use of logical relations, $\models (\phi \leftrightarrow \psi) \Rightarrow K \div \phi = K \div \psi$ translates to $\phi \equiv \psi \Rightarrow K \div \phi \equiv K \div \psi$. Applied to $k$-belief sets one has:

**K6** $G \equiv^k H \Rightarrow F \div^k G \equiv^k F \div^k H$

Figure 9 shows an illustration of two possible nearly optimal contractions. There is no optimal contraction in regards to the postulates **K1-K6** - these only know a successful contraction but it should be goal of any contraction operator implementation to do a minimal contraction in a certain sense, e. g. try to end up with a $\subseteq$-maximal AF. Two possible results of contracting $F$ with $G$ are illustrated by dashed lines, $F'$ and $F''$, their respective space of models is

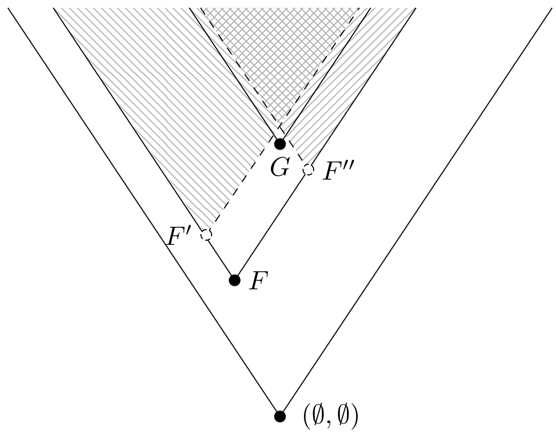



Figure 9: Dung-logic contraction

marked by the patterned areas. It is possible to move $F'$ and $F''$ "up" or "down" along the outside of $F$ but it is crucial for them to not include the models of $G$ in their space of models since this would result in an unsuccessful contraction. $F'$ and $F''$ could also be shifted more to the inner of $F$. It would however not be possible to move them out of $F$ since this would conflict with the inclusion postulate.

# 4 An Impossibility Theorem

We can show that there is no contraction operator on AFs for Dung-logics. The problem herein lies in the recovery postulate mainly. AFs have two *layers* of information, one of which is dependent on the other. Postulate **K2** demands of a contraction to never add information. This means we can only remove information of an AF when contracting. If we have to remove an argument that has attacks dependent on it which are not part of the contracted AF, we can not expand the contraction result with the contracted AF and restore all information as demanded by **K5**.

**Theorem 4.** *There is no operator* $\div^k : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ *such that* $\div^k$ *satisfies* ***K1-K6***.

*Proof.* Let us assume there *is* such a binary operator $\div^k$ and consider the AFs $F = (A, \{(a, a) \mid a \in A\})$ and $G = (A, \emptyset)$ with $A \neq \emptyset$. Obviously $F \models^k G$.

If we wanted to successfully contract $F$ with $G$, we would have to remove some elements in $A$ or $R(F)$ since $\div^k$ satisfies **K2** and therefore one has $(F \div^k G)^k \subseteq F^k$.

Since $G \neq (\emptyset, \emptyset)$ and therefore $\not\models^k G$ we also have $G^k \not\subseteq (F \div^k G)^k \subseteq F^k$ because $\div^k$ satisfies $\not\models^k G \Rightarrow F \div^k G \not\models^k G$, namely **K4**.

Because $R(G) = \emptyset$ and therefore for any $H \in \mathscr{A}$, we have $R(G) \subseteq R(H)$, our only chance of achieving **K4** lies in removing some $A' \subseteq A$ in $F$. Thus, a respective AF $F'$ not entailing $G$ can be identified by $F' = (A \backslash A', R(F) \backslash \{(a,a) \mid a \in A'\})$.

We have to remove attacks involving arguments of $A'$ in $F$ because $F'$ needs to stay an AF, i. e. it must hold that $R(F') \subseteq A(F') \times A(F')$. Otherwise we would violate postulate **K1**.

Now lets observe what holds regarding **K5** using an expansion operator as defined in definition 32. Since $\div^k$ satisfies **K5** we must end up with $F' +^k G \models^k F$ which equals $F^k \subseteq (F' +^k G)^k$. Note that $Mod(F') \cap Mod(G) \neq \emptyset$ since $G \subseteq F'$ and that $F'$ and $G$ are both $k$-r-free.

$$
\begin{aligned}
&F' +^k G && (69)\\
=&F'^k \cup G^k && \text{by definition 32} \quad (70)\\
=&F' \cup G && \text{by assumption} \quad (71)\\
=&(A \setminus A', R \setminus \{(a,a) \mid a \in A'\}) \cup G && (72)\\
=&\big((A \setminus A') \cup A, (R \setminus \{(a,a) \mid a \in A'\}) \cup \emptyset\big) && (73)\\
=&(A, R \setminus \{(a,a) \mid a \in A'\}) && (74)\\
\not\supseteq&(A, R) = F && (75)
\end{aligned}
$$

This contradicts our assumption that $\div^k$ satisfies **K1**, **K2**, **K4** and **K5**, in particular **K5**. □

To be precise, note that this proof shows that there can not be a contraction operator satisfying the postulates **K1**, **K2**, **K4** and **K5** which is a weaker proposition than our initial theorem 4. But this theorem still is a consequence of this weaker proposition and matches our essential point more accurately which is that there is no AGM-style contraction operator on Dung-logics.

# 5 Contraction Operators on Dung-logics

Although there is no contraction operator that satisfies **K1-K6** (as shown in 4), we will now present two possible contraction operators each of which violates only one postulate.

## 5.1 Naive Contraction

In this section we will define a *naive contraction* operator $\dot{-}^k$. Its name arises from the simple concept that underlies this operator. As indicated in the proof for theorem 4, for two AFs $F, G \in \mathscr{A}$ it suffices to remove some set $A' \subseteq A(G)$ from $F$ when contracting $G$ from $F$ to accomplish *success*, provided $F \models^k G$. This leaves us with the problem to decide *which* elements to remove whilst ensuring that a contraction result can be determined deterministically. Without a strict order on the set of all arguments $\mathcal{U}$ this seems to be unobtainable. The naive contraction operator avoids this problem by being *naive*, i.e. by removing every element of $G$ from $F$ when contracting $F$ with $G$. We will define this operator via an assisting operator $F - G$ that removes all arguments of $G$ in $F$.

**Definition 35.** We define an operator $- : \mathscr{A} \times 2^{\mathcal{U}} \rightarrow \mathscr{A}$:

$$(F, A) \mapsto (A(F) \setminus A, R(F) \cap (A(F) \setminus A)^2) \tag{76}$$

**Definition 36.** We define the *naive-contraction operator* $\dot{-}^k : \mathscr{A} \times \mathscr{A} \rightarrow \mathscr{A}$ as:

$$(F, G) \mapsto \begin{cases} F^k - A(G) & \text{if } F \models^k G \\ F^k & \text{else} \end{cases} \tag{77}$$

**Theorem 5.** *The naive-contraction operator $\dot{-}^k$ satisfies **K1-K4** and **K6** but does not satisfy **K5**.*

In what follows we will give proof of $\dot{-}^k$ satisfying every postulate but **K5**, i.e. *recovery*, and give a counter-example for **K5**.

**K1** $F \div^k G$ is an AF *(closure)*

**Proposition 3.** *The naive-contraction operator $\dot{-}^k$ satisfies **K1**.*

*Proof.* Both $F^k$ and $F^k - A(G)$ are AFs, therefore $F \dot{-}^k G$ is an AF. □

**K2** $F \models^k F \div^k G$ which equals $(F \div^k G)^k \subseteq F^k$ *(inclusion)*

**Proposition 4.** *The naive-contraction operator $\dot{-}^k$ satisfies **K2**.*

*Proof.* If $F \not\models^k G$, one has $F \dot{-}^k G = F^k$ and therefore obviously $F \models^k F \div^k G \equiv^k F^k$.

If however $F \models^k G$, with $A' = A(F) \setminus A(G)$ we have:

$$\begin{aligned}
& F \dot{-}^k G && \text{(78)} \\
=& F^k - A(G) && \text{by definition 36} \quad \text{(79)} \\
=& (A', R(F)^k \cap A'^2) && \text{by defintion 35} \quad \text{(80)} \\
=& (A', R(F)^k \setminus \bar{A}'^2) && \text{(81)} \\
\subseteq& (A(F), R(F)^k) = F^k && \text{(82)}
\end{aligned}$$

□

**K3** $F \not\models^k G \Rightarrow F \dot{-}^k G \equiv^k F$ *(vacuity)*

**Proposition 5.** *The naive-contraction operator satisfies **K3**.*

*Proof.* Be definition 36 we have $F \dot{-}^k G = F^k$ if $F \not\models^k G$ and therefore $F \dot{-}^k G \equiv^k F$. □

**K4** $\not\models^k G \Rightarrow F \div^k G \not\models^k G$ which equals $\not\models^k G \Rightarrow G^k \not\subseteq (F \div^k G)^k$ *(success)*

**Proposition 6.** *The naive-contraction operator* $\dot{-}^k$ *satisfies* ***K4****.*

*Proof.* We assume $F \models^k G$ and $G \neq (\emptyset, \emptyset)$ because otherwise, there is nothing to show.

Since $A(G) \neq \emptyset$ we know that $A(G) \not\subseteq A(F) \setminus A(G)$ leading to $G^k \not\subseteq (F \dot{-}^k G)^k \equiv F \dot{-}^k G \not\models^k G$. □

**K5** $(F \div^k G) +^k G \models^k F$ which equals $F^k \subseteq \big((F \div^k G) +^k G\big)^k$ *(recovery)*

**Proposition 7.** *The naive-contraction operator does not satisfy* ***K5****.*

*Proof.* A counter-example for **K5** can easily be given as we have effectively done this in section 4: Let $F = (A, \{(a, a) \mid a \in A\})$, $G = (A, \emptyset)$. Note that $F$ is $k$-r-free since kernelization never removes self-loops.

When contracting $F$ with $G$ one ends up with: $F \dot{-}^k G = (\emptyset, \emptyset)$. But $(\emptyset, \emptyset) +^k G = G$ and $G \not\models^k F$ since $G^k \not\subseteq F \subseteq F^k$. □

**Example.** In figure 10 you can see an illustration for a counter example similar to the one used in proposition 7. There, $F_{10.3}$ is the result of $F_{10.1}$ contracted with $F_{10.2}$. As you can see, expanding $F_{10.3}$ with $F_{10.2}$ would lead to $F_{10.4}$ meaning the recovery postulate can not be uphold.

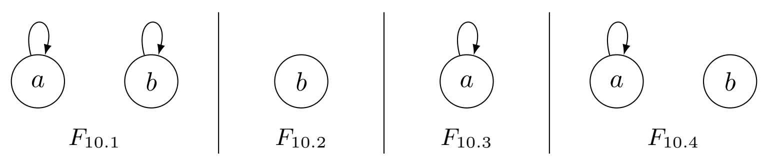



Figure 10: Counter example illustrations

**K6** $G \equiv^k H \Rightarrow F \div^k G \equiv^k F \div^k H$ *(equivalence)*

**Proposition 8.** *The naive-contraction operator* $\dot{-}^k$ *satisfies* ***K6****.*

*Proof.* For any AF $H \in \mathscr{A}$ one has $H \equiv^k G \Rightarrow A(H) = A(G)$ since kernelization only removes redundant *attacks* and no arguments. Given $F \models^k G$ one has:

$$F \dot{-}^k G \tag{83}$$

$$= F^k - A(G) \quad \text{by definition 36} \tag{84}$$

$$= F^k - A(H) \quad \text{by assumption} \tag{85}$$

$$= F \dot{-}^k H \quad \text{by definition 36} \tag{86}$$

Given $F \not\models^k G$ it applies that $F \not\models^k H$ since $H \equiv^k G$ leading to $F \dot{-}^k G = F^k = F \dot{-}^k H$. □

## 5.2 Missing Contraction

Building up on the idea of the counter-example for **K5** given in section 5.1 one might hit on the idea of not removing those arguments that hold information which is not present in the contracted AF. This should - at least on first glance - preserve *recovery*. Although we have already shown that there is no contraction operator satisfying all postulates, we might also show why this idea is doomed to fail, as well.

Given AFs $F, G \in \mathscr{A}$. The idea of arguments that hold additional information is expressed in the definition of *missing-arguments* in $G$ regarding $F$, i. e. they have some information missing in $G$ that is present in $F$.

**Definition 37.** The set of *missing-arguments* in $G$ in respects to $F$, $\mathrm{Mis}_F(G)$ is defined as:

$$\mathrm{Mis}_F(G) = \left\{ \begin{aligned} a \in A(G) \mid \exists a' \in A(F) :&(a, a') \in R(F) \setminus R(G) \\ &\vee (a', a) \in R(F) \setminus R(G) \end{aligned} \right\} \tag{87}$$

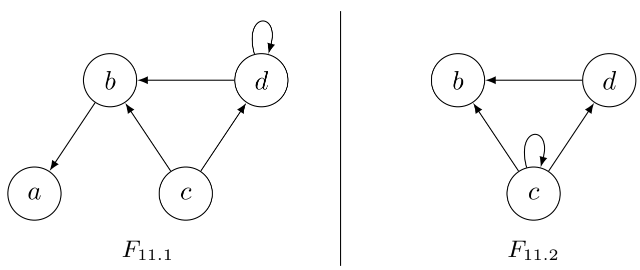



Figure 11: Missing-arguments example

**Example.** Let us look at an example for missing-arguments. In figure 11, one can see two AFs. The missing-arguments of $F_{11.2}$ in regards to $F_{11.1}$ are given by $\{b, d\}$. $b$ is a missing-argument, because $F_{11.2}$ does not contain the attack $(b, a)$. Whereas $d$ is a missing-argument, because $F_{11.2}$ misses the self-loop $(d, d)$. The self-loop $(c, c)$ which is part of $F_{11.2}$ but not $F_{11.1}$ has no influence on the set of missing-arguments because we only consider attacks missing *in* $F_{11.2}$.

Using this definition we define an operator that removes only those arguments of an AF in another AF that are involved in the same attacks in both AFs.

**Definition 38.** The operator $\setminus_{Mis} : \mathscr{A} \times \mathscr{A} \to \mathcal{U}$ is defined as:

$$(F, G) \mapsto \big(A(F) \setminus A(G)\big) \cup \mathrm{Mis}_F(G) \tag{88}$$

**Definition 39.** The *missing-contraction* operator $\hat{-}^k : \mathscr{A} \times \mathscr{A} \to \mathscr{A}$ is defined as:

$$(F, G) \mapsto \begin{cases} \big(F^k \setminus_{Mis} G^k, R(F)^k \cap (F^k \setminus_{Mis} G^k)^2\big) & \text{if } F \models^k G \\ F^k & \text{else} \end{cases} \tag{89}$$

**Theorem 6.** *The missing-contraction operator $\hat{-}^k$ satisfies all postulates **K1**-**K3** and **K5**-**K6** but does not satisfy **K4**.*

Again, we will start our elaboration of the newly introduced missing-contraction operator $\hat{-}^k$ by giving proof for the realization of all postulates but **K4**, namely *success*, for which a counter example will be given.

**K1** $F \div^k G$ is an AF *(closure)*

**Proposition 9.** *The missing-contraction operator $\hat{-}^k$ satisfies **K1**.*

*Proof.* If $F \not\models^k G$, $F \hat{-}^k G = F^k$ which is an AF.

If $F \models^k G$ one has $F \hat{-}^k G = (A, R(F)^k \cap (A \times A))$, $A = F^k \setminus_{Mis} G^k$ which obviously is an AF because $R(F)^k \cap (A \times A) \subseteq A \times A$. □

**K2** $F \models^k F \div^k G$ which equals $(F \div^k G)^k \subseteq F^k$ *(inclusion)*

**Proposition 10.** *The missing-contraction operator $\hat{-}^k$ satisfies **K2**.*

*Proof.* If $F \not\models^k G$ one has $F \hat{-}^k G = F^k \subseteq F^k$ which obviously realizes **K2**.

To give proof ot the realization of **K2** when $F \models^k G$, let us first observe what holds for $\mathrm{Mis}_F(G) \sim A(F)$, $\sim \in \{\subseteq, \supseteq\}$ in this case:

By definition 38 one has $\mathrm{Mis}_F(G) \subseteq A(G)$ and $A(G) \subseteq A(F)$ by assumption of $F \models^k G$, leading to:

$$\big(A(F) \setminus A(G)\big) \cup \mathrm{Mis}_F(G) \subseteq A(F) \cup A(G) \subseteq A(F) \tag{90}$$

From the proof of proposition 9 we know that $R(F \hat{-}^k G) \subseteq R(F)^k$. Since kernelization of $F \hat{-}^k G$ can only *remove* attacks in $R(F \hat{-}^k G)$ one has

$$R(F \hat{-}^k G)^k \subseteq R(F \hat{-}^k G) \subseteq R(F)^k \tag{91}$$

All this leads to

$$(F \hat{-}^k G)^k \subseteq F \hat{-}^k G \subseteq F^k \tag{92}$$

which implies the realization of **K2**. □

**K3** $F \not\models^k G \Rightarrow F \div^k G \equiv^k F$ *(vacuity)*

**Proposition 11.** *The missing-contraction operator $\hat{-}^k$ satisfies **K3**.*

*Proof.* **K3** is realized by definition since $F \hat{-}^k G = F^k \equiv^k F$ if $F \not\models^k G$. □

**K4** $\not\models^k G \Rightarrow F \div^k G \not\models^k G$ which equals $\not\models^k G \Rightarrow G^k \not\subseteq (F \div^k G)^k$ *(success)*

**Proposition 12.** *The missing-contraction operator $\hat{-}^k$ does not satisfy **K4**.*

*Proof.* A counter-example for **K4** is given by: $F = (A, \{(a, a) \mid a \in A\})$, $G = (A, \emptyset)$. Obivously $F \models^k G$ and $F = F^k$, $G = G^k$. Note that $\mathrm{Mis}_F(G) = A$. Thus we end up with:

$$
\begin{aligned}
& F \hat{-}^k G && (93)\\
=& \big(F^k \setminus_{Mis} G^k, R(F)^k \cap (F^k \setminus_{Mis} G^k)^2\big) \quad \text{by definition 39} && (94)\\
=& \Big((A \setminus A) \cup A, R(F)^k \cap \big((A \setminus A) \cup A\big)^2\Big) && (95)\\
=& (A, R(F)^k \cap A^2) && (96)\\
=& (A, R(F)^k) && (97)\\
\models^k & G \quad \text{by lemma 1} && (98)
\end{aligned}
$$

□

**K5** $(F \div^k G) +^k G \models^k F$ which equals $F^k \subseteq \big((F \div^k G) +^k G\big)^k$ *(recovery)*

**Proposition 13.** *The missing-contraction operator* $\hat{-}^k$ *satisfies* ***K5****.*

*Proof.* Again, we will show the realization of **K5** with a proof by cases.

First, let us assume $F \not\models^k G$, then one has $(F \hat{-}^k G) +^k G = F^k +^k G = F^k \cup G^k \subseteq F^k$ which already shows that **K5** is realized in this case.

Now let us suppose $F \models^k G$. In order to proof **K5**'s realization we must show that $A(F) \subseteq A\big((F \hat{-}^k G) +^k G\big)$ and $R(F) \subseteq R\big((F \hat{-}^k G) +^k G\big)$.

Note that by assumption, $F \models^k F \hat{-}^k G$ and lemma 1 we have $G^k \subseteq F^k$ and $(F \hat{-}^k G)^k \subseteq F^k$. This means that $G^k \cup (F \hat{-}^k G)^k$ is $k$-r-free as it is obviously bounded by $F^k$. This also means that

$$Mod^k(G) \cup Mod^k(F \hat{-}^k G) \neq \emptyset \tag{99}$$

as Baumann and Brewka [1] were able to show that $Mod^k(F) \cap Mod^k(G) \neq \emptyset \Leftrightarrow F^k \cup G^k$ is $k$-r-free.

This can be used to show that $A(F) \subseteq A\big((F \hat{-}^k G) +^k G\big)$ holds:

$$
\begin{aligned}
& A\big((F \hat{-}^k G) +^k G\big) && (100)\\
=& A\big((F \hat{-}^k G)^k \cup G^k\big) \quad \text{by (99) and definition 32} && (101)\\
=& A(F \hat{-}^k G) \cup A(G^k) \quad \text{by definition 2} && (102)\\
=& (F^k \setminus_{Mis} G^k) \cup A(G) \quad \text{by definition 39 and 18} && (103)\\
=& \Big(\big(A(F) \setminus A(G)\big) \cup \mathrm{Mis}_F(G)\Big) \cup A(G) \quad \text{by definition 38} && (104)\\
\supseteq & \big(A(F) \setminus A(G)\big) \cup A(G) = A(F) && (105)
\end{aligned}
$$

Now we must show that $R(F)^k \subseteq R\big((F \hat{-}^k G) +^k G\big)$ where by (99) $R\big((F \hat{-}^k G) +^k G\big) = R(F \hat{-}^k G)^k \cup R(G)^k$. First of all, we have:

$$R(F)^k \setminus R(F^k \hat{-}^k G^k) \subseteq R(G)^k \tag{106}$$

This can be proven indirectly: Lets assume we have a $(a, b) \in R(F)^k \setminus R(F^k \hat{-}^k G^k)$ satisfying $(a, b) \notin R(G)^k$. One has

$$
\begin{aligned}
& R(F)^k \setminus R(F \hat{-}^k G) && && (107)\\
=& R(F)^k \setminus \big(R(F^k) \cap (F^k \setminus_{Mis} G^k)^2\big) && \text{by defintion 39} && (108)\\
=& R(F)^k \setminus (F^k \setminus_{Mis} G^k)^2 && && (109)\\
=& R(F)^k \setminus \Big(\big(A(F^k) \setminus A(G^k)\big) \cup \mathrm{Mis}_{F^k}(G^k)\Big)^2 && \text{by definition 38} && (110)
\end{aligned}
$$

Alongside (110), we can conclude by our assumption that $(a, b) \notin \Big(\big(A(F^k) \setminus A(G^k)\big) \cup \mathrm{Mis}_{F^k}(G^k)\Big)^2$. Hereby we have $a, b \in A(G^k)$ as $a, b \in A(F^k)$ and $a, b \notin \mathrm{Mis}_{F^k}(G^k)$. This means $(a, b) \in R(G)^k$ because otherwise $a, b \in \mathrm{Mis}_{F^k}(G^k)$ which contradicts our assumption, thus proving (106).

Using (106) we can show the realization of **K5**:

$$
\begin{aligned}
& R(F)^k \setminus R(F^k \hat{-}^k G^k) \subseteq R(G)^k && (111)\\
\Leftrightarrow & R(F)^k \setminus R(F^k \hat{-}^k G^k) \cup R(F^k \hat{-}^k G^k) \subseteq R(G)^k \cup R(F^k \hat{-}^k G^k) && (112)\\
\Leftrightarrow & R(F)^k \subseteq R(G)^k \cup R(F^k \hat{-}^k G^k) && (113)
\end{aligned}
$$

□

**K6** $G \equiv^k H \Rightarrow F \div^k G \equiv^k F \div^k H$ *(equivalence)*

**Proposition 14.** *The missing-contraction operator* $\hat{-}^k$ *satisfies* ***K6****.*

*Proof.* Let $H \in \mathscr{A}$ such that $H \equiv^k G$, then one has $H^k = G^k$.

Given $F \not\models^k G$ one also has $F \not\models^k H$ leading to $F \hat{-}^k G = F^k = F \hat{-}^k H$.

Given $F \models^k G$ one has:

$$
\begin{aligned}
& F \hat{-}^k G && && (114)\\
=& \big(F^k \setminus_{Mis} G^k, R(F)^k \cap (F^k \setminus_{Mis} G^k)^2\big) && \text{by definition 39} && (115)\\
=& \big(F^k \setminus_{Mis} H^k, R(F)^k \cap (F^k \setminus_{Mis} H^k)^2\big) && \text{by assumption} && (116)\\
=& F \hat{-}^k H && \text{by definition 39} && (117)
\end{aligned}
$$

□

# 6 Revision and Contraction

In this section we will examine the relation of revision and contraction on Dung logics. More precisely, we will consider the revision operator $*^k$ and the naive contraction operator $\dot{-}^k$, as presented in 5.1, analyzing what holds for $(F *^k G)\dot{-}^k G$ and $(F\dot{-}^k G) *^k G$.

We will examine the result of $(F *^k G)\dot{-}^k G$ first. By definition 33 and 34 we know that there is some $I = F *^k G$ such that there is some $H \subseteq F^k$ such that $I = G^k \cup H$. Therefore we can write:

$$(F *^k G)\dot{-}^k G = (G^k \cup I)\dot{-}^k G \tag{118}$$

We know that, since $I$ is $\subseteq$-maximal, $H$ must be $\subseteq$-maximal, as well. Therefore we know that $\big(F(A) \setminus F(G), \emptyset\big) \subseteq H$ because adding arguments to $H$ will never contribute to not-$k$-r-freeness. Since $F *^k G \models^k G$ we have by lemma 1 $(F *^k G)\dot{-}^k G \subseteq F\dot{-}^k G$. This leads to

$$\big(F(A) \setminus F(G), \emptyset\big) \subseteq (F *^k G)\dot{-}^k G \subseteq F\dot{-}^k G \tag{119}$$

This result can be extended if the relation of $F \sim G$, $\sim \in \{\models^k, \not\models^k\}$ is given. Assumed $F \models^k G$ one has $F *^k G = F^k$ because $H = F^k$ leads to a $\subseteq$-maximal $I = G^k \cup F^k$. This leads to

$$\big(A(F) \setminus A(G), \emptyset\big) \subseteq (F *^k G)\dot{-}^k G = F^k \dot{-}^k G \tag{120}$$

Assumed $F \not\models^k G$ no specific upper bound can be given which leads to

$$\big(A(F) \setminus A(G), \emptyset\big) \subseteq (F *^k G)\dot{-}^k G \tag{121}$$

Now lets have a closer look at $(F\dot{-}^k G) *^k G$. Assumed $F \not\models^k G$, one has $F\dot{-}^k G = F^k$ leading to $(F\dot{-}^k G) *^k G = F^k *^k G = F *^k G$. Given $F \models^k G$, a more fine grained examination is needed. By definition (36) we have:

$$(F\dot{-}^k G) *^k G = \Big(A(F) \setminus A(G), R(F) \cap \big(A(F) \setminus A(G)\big)^2\Big) *^k G \tag{122}$$

We know that $F^k$ is $k$-r-free and since $F \models^k G$, $G^k \subseteq F^k$. Therefore $F *^k G = F^k$. This also means that $\forall F' \subseteq F^k : F' *^k G = F' \cup G^k$. Alongside with (122) we have:

$$(F\dot{-}^k G) *^k G \tag{123}$$

$$= \Big(A(F) \setminus A(G), R(F) \cap \big(A(F) \setminus A(G)\big)^2\Big) \cup G^k \tag{124}$$

$$= \Big(\big(A(F) \setminus A(G)\big) \cup A(G), R(F) \cap \big(A(F) \setminus A(G)\big)^2 \cup R(G)^k\Big) \tag{125}$$

$$= \Big(A(F), R(F) \cap \big(A(F) \setminus A(G)\big)^2 \cup R(G)^k\Big) \tag{126}$$

$$= (F\dot{-}^k G) +^k G \tag{127}$$

# 7 Conclusion

## 7.1 Results

Before we come to the actual conclusion of this thesis, let us quickly recap what we were able to achieve. We phrased three main theorems whereas at first, theorem 4 showed that there is no contraction operator on Dung-logics satisfying postulates **K1**-**K6**. We then introduced two attempts on an contraction operator for which the naive-contraction operator $\dot{-}^k$ - as theorem 5 showed - violated only postulate **K5**, namely *recovery*, whilst the missing-contraction contraction operator $\hat{-}^k$ violated only postulate **K4** which is the *success* postulate. This was shown by theorem 6.

We admit that the missing-contraction operator is not useful on its own. But it emphasizes the problem of contraction of Dung-logics being the two layers of information which make it impossible to successfully contract whilst ensuring recovery. The missing-contraction operator was an successful attempt to preserve recovery but was still unsuccessful. This operator however could be utilized in conjunction with the naive-contraction operator to create a new contraction operator with minimal AFs for which it will fail to recover.

## 7.2 The Rescue of Recovery

How can we proceed to implement an contraction operator on Dung-Logics? One could approach the problem of recovery in two ways: we could make modifications to the postulates or modifications to the logic. In what follows, we will reflect both of those options.

Let us start with an overview over possible changes to the postulates. The recovery postulate was introduced to ensure minimal change on contracted AFs. This however, as Hansson pointed out, might be a requirement that is too strong and therefore has unwanted side-effects. He was able to show in [6, p. 101] that contracting a belief set $K$ with a formerly expanded proposition $\phi$ resulted in a belief set which was $\subseteq$-greater than the original belief set as it still contained certain so-called "disjunctive residues" enabling recovery for future expansions. It therefore is not possible to "empty" a belief set up to tautologies via contractions as for every contraction those disjunctive residues would still be part of the resulting belief set. In [7, p. 259] Hansson also gave a good example for a borderline case of a contraction operator ensuring recovery and reaching a questionable result. Suppose the belief set $K = Cn(\{\phi, \psi\})$ with $\phi =$ "George is a criminal" and $\psi =$ "George is a murderer". Let us now assume we come to know that George is not a criminal at all. This would lead us to contract $K$ with $\phi$ which in turn would remove $\psi$ as $\models \psi \Rightarrow \phi$. But $\phi \Rightarrow \psi$ would still be part of the resulting belief set as only this can ensure recovery. If we now were to expand our belief set with $\chi =$ "George is a shoplifter", $\phi$ would be part of our belief set again since $\models \chi \Rightarrow \phi$. But this would lead to $\psi$ coming into our belief set again. After having contracted $K$ with $\phi$, we could not assume that George is a shoplifter without assuming him to be a murderer as well. These examples speak for why the recovery postulate is problematic in and of itself and might be dropped as a requirement for contraction operators.

There are however reasons specifically applicable to Dung-logics which speak for why the recovery postulate might be not ideal to be imposed on contraction operators. As we have already mentioned, the recovery postulate was introduced with the requirement of *minimal change* in mind. In the case of Dung-logics this goal of minimal change is not achieved. The missing-contraction operator whilst ensuring recovery does not proceed *minimal* at all. In fact, for $F \div G$ some selection function on the elements of $G$ would be more adequate to ensure a minimal contraction since it is only needed to remove *one* element of $G$ in $F$ in order to contract successfully. One might rather phrase a new postulate replacing the recovery postulate to ensure truly minimal contractions on AFs.

But when demanding, the recovery postulate should not be omitted or changed, one could alter the logic operated upon when working with AFs. We will briefly present the idea behind this logic. Since the problem with recovery were the two layers of information present in an AF, one could change the logic on AFs such that it only considers *one* layer of information which hopefully should allow for recovery and success at the same time. This could be done by limiting the logic to the attacks of an AF. Thereby, the arguments of an AF would be induced by the sets of attacks and syntactical elements of a logic succeeding this idea would be attacks. The downside of such a logic would be that isolated arguments, i. e. arguments with no attacks, could not be covered. At first glance this seems unproblematic since AFs are used to model *conflicting* pieces of knowledge, i. e. non-conflicting arguments are not of primary interest. However, in praxis isolated arguments are of interest because for systems like intelligent agents it is often needed to represent a complete system of knowledge which includes non-conflicting pieces of information as well as conflicting ones.

We can conclude this thesis with a summary of three points that have been opened up by us and which would further enhance the field of Dung-Logics in combination with abstract argumentation:

1. Can AGM-style operators be expanded to true belief *sets* of AFs?
2. Can a new postulate replacing recovery be phrased which requires minimal change on AFs when contracting?
3. Can recovery be ensured when contracting AFs for a different logic only considering attacks?

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